Largest family without $A \cup B \subseteq C \cap D$

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Abstract

Let $\mathcal{F}$ be a family of subsets of an n-element set not containing four distinct members such that $A \cup B \subseteq C \cap D$. It is proved that the maximum size of $\mathcal{F}$ under this condition is equal to the sum of the two largest binomial coefficients of order n. The maximum families are also characterized. A LYM-type inequality for such families is given, too.

Key Words: families of subsets, Sperner, LYM

1 The inequalities

Let $[n] = \{1, \ldots, n\}$ be a finite set and $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. The well-known theorem of *Sperner* ([7]) says that if no member of $\mathcal{F}$ contains another member then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality iff $\mathcal{F}$ consists of all sets of size $\lfloor n/2 \rfloor$

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or all sets of size $\lceil n/2 \rceil$. Moreover the LYM-type inequality ([5] , [8] , [6] , see also [1])

$$\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} \leq 1$$

also holds for such a family. It is easy to see that the second inequality implies the first one. On the other hand equality holds in the second inequality only when $\mathcal{F}$ consists of all sets of a fixed size.

The main aim of the present note is to investigate the analogous problem, when $\mathcal{F}$ contains no four distinct sets A, B, C, D such that A is contained in both C and D, and at the same time B is contained in both C and D. In other words,

$$\text{there are no four distinct } A, B, C, D \in \mathcal{F} \text{ with } A \cup B \subseteq C \cap D. \qquad (*)$$

It is easy to check that the family consisting of all k and $k+1$-element subsets satisfies $(*)$. We will see that this is the largest family for the appropriate choice of k.

Theorem 1 *Let $3 \leq n$. If the family $\mathcal{F} \subseteq 2^{[n]}$ satisfies $(*)$ then $|\mathcal{F}|$ cannot exceed the sum of the two largest binomial coefficients of order n, i.e., $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.*

The LYM-type inequality holds only if $\emptyset$ and $[n]$ are excluded from the family.

Theorem 2 *Let $3 \leq n$. If the family $\mathcal{F} \subseteq 2^{[n]}$ satisfies $(*)$, $\emptyset, [n] \notin \mathcal{F}$ then*

$$\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} \leq 2.$$

Let us first prove Theorem 2 by the method of cyclic permutations ([3]). Let $\{1, \ldots, n\}$ be considered as a cyclic permutation of the elements of $[n]$. That is the elements are considered $\pmod n$. An *interval* is a subset of form $\{k, k+1, \ldots, l\}$ where $1 \leq k, l \leq n$. Intervals will be denoted by $\hat{A}, \hat{B}$ etc. Families of intervals are denoted by $\hat{\mathcal{A}}, \hat{\mathcal{B}}$, etc. The proof starts with two lemmas.

Lemma 1 *Let $\hat{\mathcal{F}}$ be a family of intervals such that any member $\hat{F} \in \hat{\mathcal{F}}$ is contained in at most one other member of $\hat{\mathcal{F}}$, furthermore $\emptyset, [n] \notin \hat{\mathcal{F}}$. If m denotes the number of the maximal members, a denotes the number of non-maximal members then*

$$m + \frac{a}{2} \leq n \qquad (1)$$

holds.

Proof. A *chain* is a family of subsets $L_1 \subset \cdots \subset L_n$ where $|L_i| = i (1 \leq i \leq n)$. The number of chains of intervals containing $\hat{F}$ is $2^{|\hat{F}|-1} 2^{n-|\hat{F}|-1} = 2^{n-2}$.

Suppose that $\hat{A} \subset \hat{B}, \hat{A} \neq \hat{B}$. We give an upper bound on the number of chains containing both of them. The number of choices of the new members of the chains "between the two sets" is at most $2^{|\hat{B}|-|\hat{A}|-1}$ since, at least once, there is only one choice. Therefore the number of such chains is at most $2^{|\hat{A}|-1}2^{|\hat{B}|-|\hat{A}|-1}2^{n-|\hat{B}|-1} = 2^{n-3}$. The total number of chains is $n2^{n-2}$. Since a chain contains one or two members, we obtain the inequality

$$(m+a)2^{n-2} \leq n2^{n-2} + a2^{n-3}$$

which is equivalent to the the statement of the lemma. ■

Lemma 2 *If $\hat{\mathcal{F}}$ is a family of intervals satisfying $(*)$, and $\emptyset, [n] \notin \hat{\mathcal{F}}$, then $|\hat{\mathcal{F}}| \leq 2n$.*

Proof. It is easy to see by complementation that the previous lemma holds for a family in which any member contains at most one other member. Divide $\hat{\mathcal{F}}$ into three subfamilies: the maximal $(\hat{\mathcal{M}}_1)$, the minimal $(\hat{\mathcal{M}}_2)$ and other members $(\hat{\mathcal{A}})$. Introduce the notations $|\hat{\mathcal{M}}_1| = m_1, |\hat{\mathcal{M}}_2| = m_2, |\hat{\mathcal{A}}| = a$. It is easy to see that $(*)$ implies that $\hat{\mathcal{M}}_1 \cup \hat{\mathcal{A}}$ satisfies the conditions of the previous lemma. Therefore we have $m_1 + \frac{a}{2} \leq n$. On the other hand $\hat{\mathcal{M}}_2 \cup \hat{\mathcal{A}}$ satisfies the complementing of the previous lemma, we obtain the inequality $m_2 + \frac{a}{2} \leq n$. The sum of the two inequalities is $m_1 + m_2 + a \leq 2n$ as desired. ■

Proof of Theorem 2. We will double-count the pairs $(\mathcal{C}, F)$ where $\mathcal{C}$ is a cyclic permutation of $[n]$, $F \in \mathcal{F}$ and F is an interval along $\mathcal{C}$. For a fixed F the number of cyclic permutations is $|F|!(n-|F|)!$ therefore the number of pairs in question is

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)!.$$

For a fixed cyclic permutation $\mathcal{C}$ the number of possible Fs is at most $2n$ by the previous lemma. We obtained the the inequality

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq (n-1)!2n.$$

This is equivalent to the statement of the theorem. ■

Proof of Theorem 1. If none of $\emptyset$ and $[n]$ is a member of $\mathcal{F}$ then the statement is an easy consequence of Theorem 2. If both of them are in $\mathcal{F}$ then $\mathcal{F} - \{\emptyset, [n]\}$ is a Sperner family, therefore we have the upper estimate $\binom{n}{\lfloor n/2 \rfloor} + 2$, which is less than our need, if $3 \leq n$. Suppose that exactly one of $\emptyset$ and $[n]$ is in $\mathcal{F}$. By complementation $\emptyset \in \mathcal{F}$ can be supposed. Then $\mathcal{F}' = \mathcal{F} - \{\emptyset\}$ contains no 3 distinct members A, B, C such that $A \subset B, A \subset C$. It was proved in [4] (our Corollary 2 in Section 2 is slightly weaker) that

$$|\mathcal{F}'| \leq \left(1 + \frac{2}{n}\right) \binom{n}{\lfloor n/2 \rfloor}$$

holds under this condition. This upper estimate is strong enough when $3 \leq n$. ■

Remark. Dániel Gerbner (student in Budapest) [2] noticed that there is no need to use the theorem from [4], since replacing $\emptyset$ by an arbitrarily chosen one-element set $\{i\} \notin \mathcal{F}$ reduces the problem to the case when $\emptyset, [n] \notin \mathcal{F}$. The case when $\emptyset$ and all one-element sets are in $\mathcal{F}$ is trivial.

2 Cases of equality

The methods of the previous section are not strong enough for finding the cases of equality. The conditions of Lemma 1 allow a large variety of families with equality. Therefore we have to consider the whole original family, rather than just the intervals. An *antichain* is a family of sets containg no comparable members.

Lemma 3 *Let $\mathcal{M}$ and $\mathcal{A}$ be two disjoint antichains in $2^{[n]}$ where $[n] \notin \mathcal{M}$. Suppose that for any $A \in \mathcal{A}$ there is a unique $f(A) \in \mathcal{M}$ with $A \subset f(A)$. Then*

$$\sum_{M\in\mathcal{M}} \binom{n}{|M|}^{-1} + \sum_{A\in\mathcal{A}} \binom{n}{|A|}^{-1} \left(1 - \frac{1}{n-|A|}\right) \leq 1 \tag{2}$$

holds, with equality only when either $|f(A)| = n-1$ or $|f(A)| = |A|-1$ holds for each $A \in \mathcal{A}$.

Proof. The number of chains containing a set M is $|M|!(n-|M|)!$. Adding these numbers for all members of $\mathcal{M}$ and $\mathcal{A}$, a chain is counted once or twice, the latter can happen only if the chain contains an $A \in \mathcal{A}$ and $f(A) \in M$. The total number of chains is $n!$, the number of chains containing both A and $f(A)$ is $|A|!(|f(A)|-|A|)!(n-|f(A)|)!$. Hence we have the following inequality:

$$\sum_{M\in\mathcal{M}} |M|!(n-|M|)! + \sum_{A\in\mathcal{A}} |A|!(n-|A|)! \leq n! + \sum_{A\in\mathcal{A}} |A|!(|f(A)|-|A|)!(n-|f(A)|)!.$$

Dividing by $n!$ we obtain

$$\sum_{M\in\mathcal{M}} \binom{n}{|M|}^{-1} + \sum_{A\in\mathcal{A}} \binom{n}{|A|}^{-1} \left(1 - \binom{n-|A|}{n-|f(A)|}^{-1}\right) \leq 1. \tag{3}$$

Since $|A| < |f(A)| < n$, the inequality $n-|A| \leq \binom{n-|A|}{n-|f(A)|}$ can be used in (3) to obtain (2). ∎

We know that $2 \leq n-|A|$, which implies the following immediate corollary.

Corollary 1 *Under the conditions of Lemma 3*

$$\sum_{M\in\mathcal{M}} \binom{n}{|M|}^{-1} + \sum_{A\in\mathcal{A}} \frac{1}{2}\binom{n}{|A|}^{-1} \leq 1 \tag{4}$$

holds, with equality only when $|A| = n-2, |f(A)| = n-1$ for each $A \in \mathcal{A}$.

Corollary 2 [4] *Let $4 \leq n$. Suppose that the family $\mathcal{F}$ contains no three distinct members A, B, C such that $A \subset B, C$. Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n-3}\right) \tag{5}$$

holds.

Proof. If $[n] \in \mathcal{F}$ then the rest of $\mathcal{F}$ satisfies the conditions of the Sperner theorem, therefore we can suppose $[n] \notin \mathcal{F}$. If we see that

$$\binom{n}{|A|} \frac{n-|A|}{n-|A|-1} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n-3}\right)$$

holds for every $0 \leq |A| \leq n-2$ then Lemma 3 implies (5). That is, we have to find the maximum of the function $g(i) = \binom{n}{i} \frac{n-i}{n-i-1}$ in the interval $0 \leq i \leq n-2$. Here $g(i-1) \leq g(i)$ holds if and only if $i(n-i-1) \leq (n-i)^2$. The discriminant $\sqrt{n^2-6n+1}$ of this quadratic inequality can be bounded from below and above by $n-4$ and $n-3$, respectively, provided $7.5 < n$. Hence $g(i-1) < g(i)$ holds if and only if $1 \leq i < \frac{n}{2}+1$. The function $g(i)$ takes on its maximum in the interval $1 \leq i \leq n-2$ at $\lfloor \frac{n+1}{2} \rfloor$. The cases $n = 4, 5, 6, 7$ can be checked separately. ∎

This corollary is slightly weaker than the statement in [4] , but its proof is much shorter.

Theorem 3 *If $5 \leq n$ then the equality in Theorems 1 and 2 implies that the family consist all k and $k+1$-element sets for some k. For $n = 3$ this is true only for Theorem 2, in Theorem 1 there are other extremal constructions: the family*

$$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,3\}\}$$

and its isomorphic versions. For $n = 4$ there is up to isomorphism one more extremal family for both theorems:

$$\binom{[4]}{2} \cup \{\{1\}, \{2,3,4\}, \{2\}, \{1,3,4\}\}.$$

Proof. Similarly to the proof of Lemma 2 define $\mathcal{M}_1$ and $\mathcal{M}_2$ as the families of maximal and minimal members of $\mathcal{F}$, respectively. $\mathcal{A} = \mathcal{F} - \mathcal{M}_1 - \mathcal{M}_2$. It is easy to see that $\mathcal{M}_1 \cup \mathcal{A}$ satisfies the conditions of Corollary 1. On the other hand, the complements of the members of $\mathcal{M}_2 \cup \mathcal{A}$ also satisfy it. The sum of the two inequalities again yield the statement of Theorem 2. If $4 < n$ there is no A satisfying the conditions of equality in Corollary 1 for both (direct and complementing) cases. Therefore in this case the equality in Theorem 3 implies $\mathcal{A} = \emptyset$. It is well-known that $\mathcal{F}$ may consist of two full levels, only. The cases $n = 3, 4$ can be checked separately. ∎

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